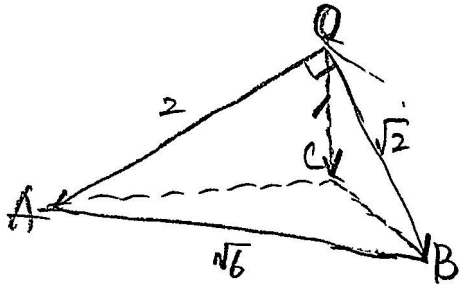


$$OA = 2, OB = \sqrt{2}, OC = 1 \quad \angle AOB = 90^\circ$$

$$\angle AOC = 60^\circ$$

$$\angle BOC = 45^\circ$$



$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \angle AOB = 2 \cdot \sqrt{2} \cdot \cos 90^\circ = 0.$$

$$\vec{b} \cdot \vec{c} = |\vec{b}| |\vec{c}| \cos \angle BOC = \sqrt{2} \cdot 1 \cdot \cos 45^\circ = 1.$$

$$\vec{c} \cdot \vec{a} = |\vec{c}| |\vec{a}| \cos \angle AOC = 1 \cdot 2 \cdot \cos 60^\circ = 1.$$

(1)

$$\vec{OH} = s\vec{b} + t\vec{c}$$

$$\vec{AH} \perp \triangle OBC \iff$$

$$\vec{AH} \perp \vec{b} \quad \text{かつ} \quad \vec{AH} \perp \vec{c}$$

$$\vec{AH} = \vec{OH} - \vec{OA}$$

$$= -\vec{a} + s\vec{b} + t\vec{c}$$

$$\vec{AH} \cdot \vec{b} = (-\vec{a} + s\vec{b} + t\vec{c}) \cdot \vec{b}$$

$$= \underbrace{-\vec{a} \cdot \vec{b}}_0 + s \underbrace{|\vec{b}|^2}_2 + t \underbrace{\vec{b} \cdot \vec{c}}_1 = 0.$$

$$2s + t = 0 \quad \dots \textcircled{1}$$

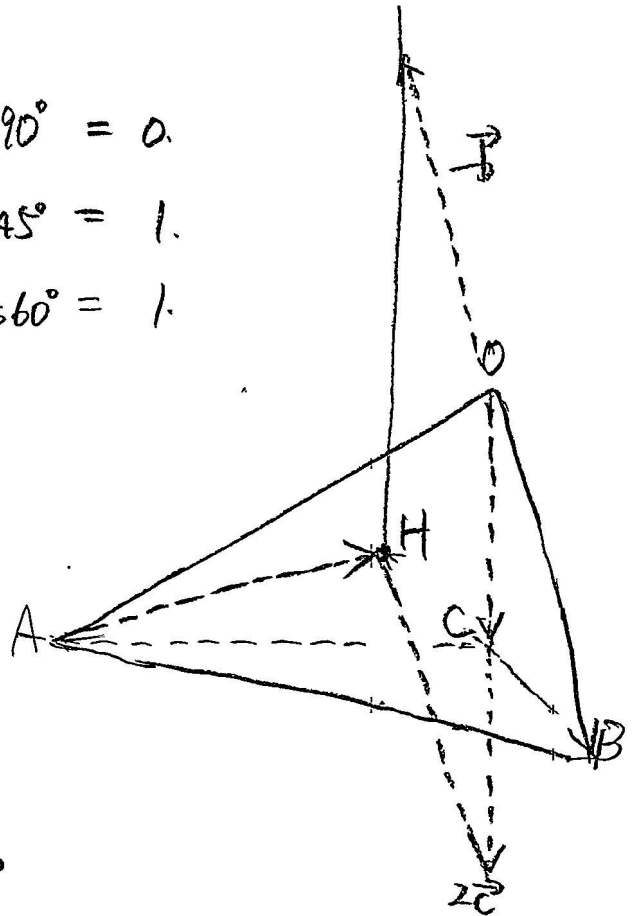
$$\vec{AH} \cdot \vec{c} = (-\vec{a} + s\vec{b} + t\vec{c}) \cdot \vec{c}$$

$$= \underbrace{-\vec{a} \cdot \vec{c}}_1 + s \underbrace{\vec{b} \cdot \vec{c}}_1 + t \underbrace{|\vec{c}|^2}_1 = 0$$

$$s + t = 1 \quad \dots \textcircled{2}$$

$$\textcircled{1} \textcircled{2} \neq \text{I} \quad s = -1, t = 2.$$

$$\therefore \vec{OH} = -\vec{b} + 2\vec{c} \quad \text{点Hは}\triangle OBC\text{の外部}$$

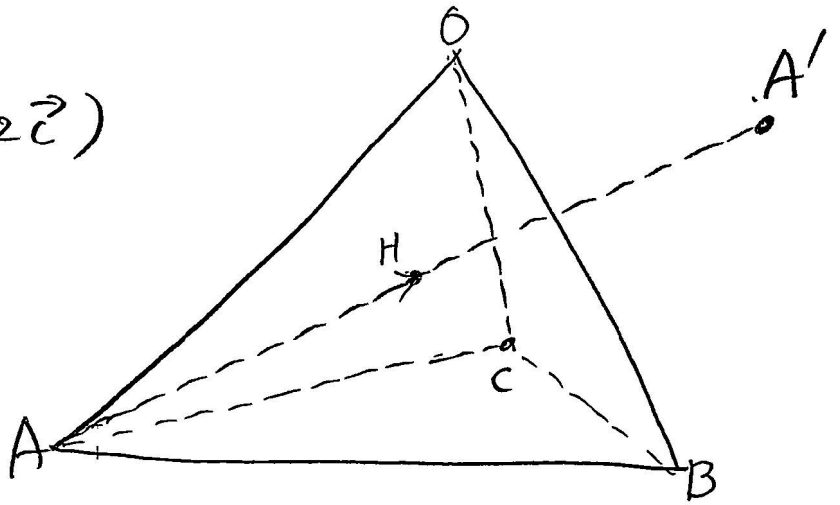


(2)

$$\vec{AA'} = 2\vec{AH}$$

$$\vec{OA'} - \vec{OA} = 2(-\vec{a} - \vec{b} + 2\vec{c})$$

$$\vec{OA'} = -\vec{a} - 2\vec{b} + 4\vec{c}$$



$$\vec{OG} = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$$

G is the centroid of $\triangle ABC$. P_0 is on AA' such that $AP_0 = u$ and $P_0A' = 1-u$.

$$\vec{OP_0} = u\vec{OA'} + (1-u)\vec{OG}$$

$$= u(-\vec{a} - 2\vec{b} + 4\vec{c})$$

$$+ (1-u) \cdot \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$$

$$= \left(-\frac{4}{3}u + \frac{1}{3}\right)\vec{a} + \left(-\frac{7}{3}u + \frac{1}{3}\right)\vec{b}$$

$$+ \left(\frac{11}{3}u + \frac{1}{3}\right)\vec{c}$$

Since P_0 is a point on the plane OBC ,

$$\text{the coefficient of } \vec{a} \text{ is } 0, \text{ so } -\frac{4}{3}u + \frac{1}{3} = 0.$$

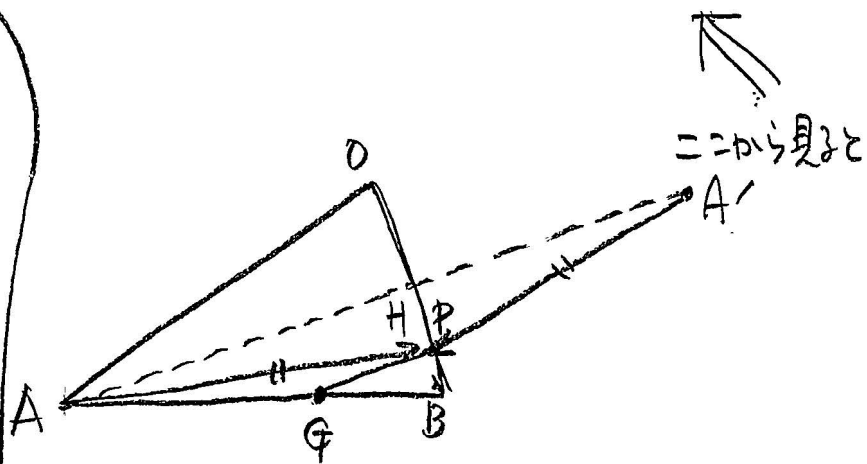
For this value of u , P_0 is not on the plane OBC but is on the line AG .

$$\therefore u = \frac{1}{4}$$

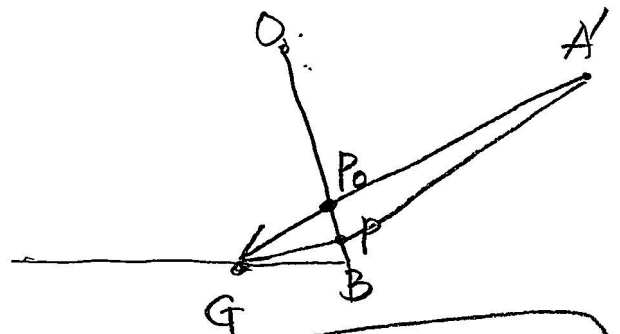
$$\therefore \vec{OP_0} = -\frac{1}{4}\vec{b} + \frac{5}{4}\vec{c}$$

P_0 is the intersection of AG and $\triangle OAB$.

$$\begin{aligned} \text{Minimum value of } AP + PG &= |\vec{AG}| = |\vec{OG} - \vec{OA}| = \left| \frac{1}{3}(\vec{a} + \vec{b} + \vec{c}) - (-\vec{a} - 2\vec{b} + 4\vec{c}) \right| \\ &= \left| \frac{4}{3}\vec{a} + \frac{7}{3}\vec{b} - \frac{11}{3}\vec{c} \right| = \frac{1}{3} |4\vec{a} + 7\vec{b} - 11\vec{c}| \end{aligned}$$



$$AP + PG = A'P + PG \geq A'G$$



$A'PG$ is a straight line segment perpendicular to the plane OBC when $AP + PG$ is minimized.

ここで

$$\begin{aligned} |4\vec{a} + 7\vec{b} - 11\vec{c}|^2 &= 16|\vec{a}|^2 + 49|\vec{b}|^2 + 121|\vec{c}|^2 \\ &\quad + 56\vec{a}\cdot\vec{b} - 154\vec{b}\cdot\vec{c} - 88\vec{c}\cdot\vec{a} \\ &= 16 \times 4 + 49 \times 2 + 121 \times 1 - 154 - 88 \\ &= 64 + 98 + 121 - 154 - 88 \\ &= 41 \end{aligned}$$

$$\therefore |4\vec{a} + 7\vec{b} - 11\vec{c}| = \sqrt{41}$$

よって $AP + PG$ の最小値は $\frac{\sqrt{41}}{3}$